THE AVERAGE NUMBER OF INTEGRAL POINTS IN ORBITS

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Dedicated to Joseph H. Silverman on the occasion of his 60*th birthday.*

ABSTRACT. Over a number field K, a celebrated result of Silverman states that if $\phi \in K(x)$ is a rational function whose second iterate is not a polynomial, the set of S-integral points in the orbit $\mathcal{O}_\phi(b) = \{\phi^n(b)\}_{n\geqslant 0}$ is finite for all $b \in \mathbb{P}^1(K)$. In this paper, we show that if we vary ϕ and b in a suitable family, the number of S-integral points of $\mathcal{O}_\phi(b)$ is absolutely bounded. In particular, if we fix $\phi \in K(x)$ and vary the base point $b \in \mathbb{P}^1(K)$, we show that $\#(\mathcal{O}_\phi(b) \cap \mathcal{O}_{K,S})$ is zero on average. Finally, we prove a zero-average result in general, assuming a standard height uniformity conjecture in arithmetic geometry, and prove it unconditionally over global function fields.

Introduction

Let $K/\mathbb{Q}$ be a number field, let S be a finite set of places (including all of the archimedean ones), and let φ^n denote the n^{th} iterate of φ. If φ^2 is not a polynomial, Silverman proved in [26] that the forward orbit

$$\mathcal{O}_\varphi(b) := \{b, \varphi(b), \varphi^2(b), \dots\} \tag{1}$$

contains only finitely many S-integral points for all $b \in \mathbb{P}^1(K)$. Moreover, Hsia and Silverman [8] gave an explicit bound on the number of S-integral points in $\mathcal{O}_\varphi(b)$, though it is normally much larger than the actual number. Nevertheless, there are rational maps (of every degree) with arbitrarily many integral points, illustrating the problem's subtlety [24, Prop. 3.46].

On the other hand, one may hope to control $\#(\mathcal{O}_\varphi(b) \cap \mathcal{O}_{K,S})$ on average, that is, as we vary over $b \in \mathbb{P}^1(K)$. This point of view has yielded some powerful insight in other areas of number theory [1, 2, 17], and we proceed with this approach here.

We begin by fixing some notation. A rational map $\varphi : \mathbb{P}^1 \to \mathbb{P}^1$ of degree d is given by two homogenous polynomials

$$\varphi = [F, G] = [a_dX^d + a_{d-1}X^{d-1}Y + \cdots + a_0Y^d, b_dX^d + b_{d-1}X^{d-1}Y + \cdots + b_0Y^d]$$

such that the resultant $\mathrm{Res}(F, G) \neq 0$. In this way, a rational map is determined by a $(2d+2)$-tuple of numbers $(a_0, a_1, \dots, a_d, b_0, b_1, \dots b_d)$, well-defined up to scaling. In particular, we may identify the set of rational maps of degree d, denoted Rat_d, as an open subset of $\mathbb{P}^{2d+1}$; see [24, §4.3]. Similarly, we define the height of φ, written $h(\varphi)$, to be its corresponding height in projective space [24, §3.1].

In this paper, we consider integral points in families $\phi : X \to \mathrm{Rat}_d$ of dynamical systems, where $X_{/K}$ is a projective variety and ϕ is a rational map defined over K. Specifically, if X is equipped with a morphism $\beta : X \to \mathbb{P}^1$, then we study $(\mathcal{O}_{\phi_P}(\beta_P) \cap \mathcal{O}_{K,S})$, given by evaluating ϕ and β at suitable points $P \in X(K)$.

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To make this explicit, we define the following subset of X:

$$I_{X,\phi} := \big\{P \in X \mid \phi_P \in \mathrm{Rat}_d \text{ is defined, and } \phi_P^2 \notin \bar{K}[x]\big\}. \tag{2}$$

When X is a curve, we prove that the quantity $\#(\mathcal{O}_{\phi_P}(\beta_P) \cap \mathcal{O}_{K,S})$ is uniformly bounded over all points $P \in I_{X,\phi}(K)$ and all morphisms $\beta : X \to \mathbb{P}^1$ of sufficiently high degree. In particular, given the relative freedom we have in choosing the basepoint family β, we have made progress towards a dynamical analog [24, Conjecture 3.47] of a conjecture of Lang [14, page 140] regarding the number of integral points on elliptic curves.

Theorem 1.1. *Let $\phi : X \to \mathrm{Rat}_d \subseteq \mathbb{P}^{2d+1}$ and $\beta : X \to \mathbb{P}^1$ be rational maps over K. If*

$$\deg(\beta) > \frac{2d-1}{d-1} \cdot \deg(\phi^* H), \tag{3}$$

for the hyperplane class $H \in \mathrm{Pic}(\mathbb{P}^{2d+1})$, then $\#\big(\mathcal{O}_{\phi_P}(\beta_P) \cap \mathcal{O}_{K,S}\big)$ is uniformly bounded over all points $P \in I_{X,\phi}(K)$.

Now to define a suitable notion of average. Given an ample height function h_X on X and a positive real number B, we write $I_{X,\phi}(K, B)$ for the set of points in $X(K) \cap I_{X,\phi}$ of height at most B. Moreover, since we pay particular attention to the case when X is a curve, we take h_X to be the height function associated to $\beta : X \to \mathbb{P}^1$.

Finally, we define the *average number of S-integral points in orbits in the family* (X, ϕ, β) to be

$$\overline{\mathrm{Avg}}(\phi, \beta, S) := \limsup_{B \to \infty} \frac{\sum_{P \in I_{X,\phi}(B,K)} \#\big(\mathcal{O}_{\phi_P}(\beta_P) \cap \mathcal{O}_{K,S}\big)}{\# I_{X,\phi}(B, K)}, \tag{4}$$

and we show that $\overline{\mathrm{Avg}}(\phi, \beta, S) = 0$ for several one-dimensional families (X, ϕ, β), including the motivating case of constant families: $\phi : X \to \mathrm{Rat}_d$ and $\phi_P = \varphi$ for all $P \in X$.

Theorem 1.2. *Let $\varphi \in K(x)$ be such that $\deg(\varphi) \geqslant 2$ and $\varphi^2 \notin \bar{K}[x]$. If X is a curve of genus $g \geqslant 1$ and $\beta : X \to \mathbb{P}^1$ is non-constant, then the set*

$$\{P \in X(K) \mid \big(\mathcal{O}_\varphi(\beta_P) \cap \mathcal{O}_{K,S}\big) \neq \varnothing\}$$

is finite. Moreover, if X is a curve with infinitely many K-rational points (necessarily $g \leqslant 1$), then

$$\overline{\mathrm{Avg}}(\varphi, \beta, S) := \limsup_{B \to \infty} \frac{\sum_{P \in I_{X,\varphi}(B,K)} \#\big(\mathcal{O}_\varphi(\beta_P) \cap \mathcal{O}_{K,S}\big)}{\# I_{X,\varphi}(B, K)} = 0. \tag{5}$$

For examples of non-constant, one-parameter families (X, ϕ, β) satisfying $\overline{\mathrm{Avg}}(\phi, \beta, S) = 0$, see Theorem 2.1. Furthermore, we prove in Theorem 3.1 that $\overline{\mathrm{Avg}}(\phi, \beta, S) = 0$ for all curves X, assuming a standard height uniformity conjecture in arithmetic geometry; this result may be viewed as a strengthened, average Dynamical-Lang Conjecture [24, Conjecture 3.47]. Moreover, we outline a generalization to varieties of arbitrary dimension, including an explicit 3-dimensional family in Theorem 3.2. Finally, when $K/\mathbb{F}_q(t)$ is global function field, we prove that $\overline{\mathrm{Avg}}(\phi, \beta, S) = 0$ for families $\phi : X \to \mathrm{Rat}_d$ acting periodically at infinity; see Theorem 4.1.

Integral points in orbits in families

Throughout this section, let X be curve and let K be a number field. Before we state and prove our averaging results, we use some properties of height functions on X to prove that the quantity $\#(\mathcal{O}_{\phi_P}(\beta_P) \cap \mathcal{O}_{K,S})$ is bounded for all morphisms $\beta : X \to \mathbb{P}^1$ of sufficiently high degree (restated from the introduction). In particular, it follows that $\overline{\mathrm{Avg}}(\phi, \beta, S)$ is bounded.

Theorem 1.1. *Let $\phi : X \to \mathrm{Rat}_d \subseteq \mathbb{P}^{2d+1}$ and $\beta : X \to \mathbb{P}^1$ be rational maps over K. If*

$$\deg(\beta) > \frac{2d-1}{d-1} \cdot \deg(\phi^* H), \tag{3}$$

for the hyperplane class $H \in \mathrm{Pic}(\mathbb{P}^{2d+1})$, then $\#\big(\mathcal{O}_{\phi_P}(\beta_P) \cap \mathcal{O}_{K,S}\big)$ is uniformly bounded over all points $P \in I_{X,\phi}(K)$.

Moreover, in the case $X = \mathbb{P}^1$, we can make Theorem 1.1 completely explicit.

Corollary 2.1. *For functions $a_i, b_j, \beta \in K(t)$, define a rational map $\phi : \mathbb{P}^1 \to \mathrm{Rat}_d$ corresponding to the family*

$$\phi_t(x) := \frac{a_d(t)x^d + a_{d-1}(t)x^{d-1} + \dots a_0(t)}{b_d(t)x^d + b_{d-1}(t)x^{d-1} + \dots b_0(t)}.$$

If $\phi^2 \notin K(t)[x]$, when we view ϕ as an element of $K(t)(x)$, and

$$\deg(\beta) > \frac{4d^2 + 2d - 2}{d-1} \cdot \max\{\deg(a_i), \deg(b_j)\},$$

then $\#\big(\mathcal{O}_{\phi_P}(\beta_P) \cap \mathcal{O}_{K,S}\big)$ is uniformly bounded over all $P \in I_{\mathbb{P}^1,\phi}(K)$, and $I_{\mathbb{P}^1,\phi}(K)$ is infinite.

To prove Theorem 1.1, we need several elementary height estimates; for a nice introduction to heights and canonical heights in arithmetic geometry, see [7, Part B] and [24, Chapter 3].

Proposition 2.1. *Let $\varphi : \mathbb{P}^1 \to \mathbb{P}^1$ be a rational map of degree $d \geqslant 2$ defined over K. There is a constant c_d, depending only on d, such that the following estimates hold for all $P \in \mathbb{P}^1(\bar{K})$.*

(1) $\big|h(\varphi(P)) - dh(P)\big| \leqslant (2d-1) \cdot h(\varphi) + c_d$.

(2) $\big|\hat{h}_\varphi(P) - h(P)\big| \leqslant (2d-1)/(d-1) \cdot h(\varphi) + c_d/(d-1)$.

(3) $\hat{h}_\varphi(\varphi(P)) = d\hat{h}_\varphi(P)$

(4) $P \in \mathrm{PrePer}(\varphi)$ *if and only if* $\hat{h}_\varphi(P) = 0$.

Proof. See, for example, [7, §§B.2,B.4] or [24, §3.4 and Exercise 3.8]. □

(Proof of Theorem 1.1). As the bound in [8, Corollary 17] suggests, we must give a lower bound on the quantity $\hat{h}_{\phi_P}(\beta_P)$ as we range over suitable points $P \in I_{X,\phi}(K)$. However, part (2) of Proposition 2.1 implies that

$$\hat{h}_{\phi_P}(\beta_P) \geqslant h_{\mathbb{P}^1}(\beta_P) - (2d-1)/(d-1) \cdot h_{\mathbb{P}^{2d+1}}(\phi_P) - c_d/(d-1) \tag{6}$$

for all $P \in I_{X,\phi}(K)$. On the other hand, let H_1 and H_2 be hyperplane classes in $\mathrm{Pic}(\mathbb{P}^1)$ and $\mathrm{Pic}(\mathbb{P}^{2d+1})$ respectively. Then $h_{\mathbb{P}^1,H_1}(Q_1) = h_{\mathbb{P}^1}(Q_1) + O(1)$ for all points $Q_1 \in \mathbb{P}^1(\bar{K})$, and $h_{\mathbb{P}^{2d+1},H_2}(Q_2) = h_{\mathbb{P}^{2d+1}}(Q_2) + O(1)$ for all $Q_2 \in \mathbb{P}^{2d+1}(\bar{K})$; see [23, Theorem 10.1(a)].

Similarly, if we extend $\beta : X \to \mathbb{P}^1$ and $\phi : X \to \mathbb{P}^{2d+1}$ to morphisms [25, II.2 Prop 2.1], the functoriality of heights [23, Theorem 10.1(d)] implies that $h_{X,\beta^*H_1}(P) = h_{\mathbb{P}^1,H_1}(\beta_P) + O(1)$ and $h_{X,\phi^*H_2}(P) = h_{\mathbb{P}^{2d+1},H_2}(\phi_P) + O(1)$ for all $P \in X(\bar{K})$. In particular, after combining these observations with (6), we deduce that there exists an absolute constant $C(d) > 0$, depending on d, such that

$$\hat{h}_{\phi_P}(\beta_P) \geqslant h_{X,\beta^*H_1}(P) - (2d-1)/(d-1) \cdot h_{X,\phi^*H_2}(P) - C(d) \tag{7}$$

for all $P \in I_{X,\phi}(K)$. However, [23, Theorem III.10.2] implies that

$$\lim_{h_{X,\beta^*H_1}(P)\to\infty} \frac{h_{X,\phi^*H_2}(P)}{h_{X,\beta^*H_1}(P)} = \frac{\deg(\phi^*H_2)}{\deg(\beta^*H_1)} \qquad \text{for all} \quad P \in X(\bar{K}). \tag{8}$$

Hence, for all $\epsilon > 0$, there exists $\delta = \delta(\epsilon)$ such that

$$\hat{h}_{\phi_P}(\beta_P) \geqslant \left(1 - \frac{2d-1}{d-1}\cdot\frac{\deg(\phi^*H_2)}{\deg(\beta^*H_1)} - \frac{2d-1}{d-1}\cdot\epsilon\right) h_{X,\beta^*H_1}(P) - C(d)$$

for all $P \in I_{X,\phi}(K)$ satisfying $h_{X,\beta^*H_1}(P) \geqslant \delta$. Therefore, if we fix an ϵ in the range

$$0 < \epsilon < (d-1)/(2d-1) - \deg(\phi^*H_2)/\deg(\beta^*H_1), \tag{9}$$

possible by assumption on $\deg(\beta) = \deg(\beta^*H_1)$, then there exists $\lambda(\epsilon) > 0$ such that

$$\boxed{\hat{h}_{\phi_P}(\beta_P) \geqslant \lambda(\epsilon)\cdot h_{X,\beta^*H_1}(P) - C(d), \quad \text{for all} \quad P \in I_{X,\phi}(K).} \tag{10}$$

In particular, for ϵ as on (9), if $P \in I_{X,\phi}(K)$ and $h_{X,\beta^*H_1}(P) \geqslant \max\left\{\delta(\epsilon), \frac{C(d)}{\lambda(\epsilon)}\right\}$, then the right hand side of (10) and $\hat{h}_{\phi_P}(\beta_P)$ are positive. From this point forward, we assume these conditions hold.

We now estimate the number of S-integral points. Specifically, [8, Corollary 17] implies that there is a constant $\gamma = \gamma\big(d, [K:\mathbb{Q}]\big)$ such that

$$\#\big(\mathcal{O}_{\phi_P}(\beta_P) \cap \mathcal{O}_{K,S}\big) \leqslant 4^{\#S}\cdot\gamma + \log_d^+\left(\frac{h_{\mathbb{P}^{2d+1}}(\phi_P)}{\hat{h}_{\phi_P}(\beta_P)}\right). \tag{11}$$

On the other hand, we see that (8) and the fact that $h_{X,\phi^*H_2}(P) = h_{\mathbb{P}^{2d+1}}(\phi_P) + O(1)$ imply

$$h_{\mathbb{P}^{2d+1}}(\phi_P) \leqslant \left(\frac{\deg(\phi^*H_2)}{\deg(\beta^*H_1)} + \epsilon\right)\cdot h_{X,\beta^*H_1}(P) + C(d), \tag{12}$$

whenever $h_{X,\beta^*H_1}(P) > \delta(\epsilon)$. Therefore, (10) and (12) yield the following statement:

$$h_{X,\beta^*H_1}(P) \geqslant \max\left\{\delta(\epsilon), \frac{C(d)}{\lambda(\epsilon)}\right\} \quad \text{implies} \quad \frac{h_{\mathbb{P}^{2d+1}}(\phi_P)}{\hat{h}_{\phi_P}(\beta_P)} \leqslant \frac{\left(\frac{\deg(\phi^*H_2)}{\deg(\beta^*H_1)} + \epsilon\right)\cdot h_{X,\beta^*H_1}(P) + C(d)}{\lambda(\epsilon)\cdot h_{X,\beta^*H_1}(P) - C(d)}.$$

However, as a real valued function, any linear fractional transformation $\rho(x) = \frac{ax+b}{cx+d}$ is bounded away from its poles. Hence, if we view $h_{X,\beta*H_1}(P)$ as the variable x, we see that (11) implies that there exists a constant $M_{\phi,\beta,\epsilon,S}$ such that $\#\big(\mathcal{O}_{\phi_P}(\beta_P) \cap \mathcal{O}_{K,S}\big) \leqslant M_{\phi,\beta,\epsilon,S}$ for all $P \in I_X(K)$ satisfying $h_{X,\beta^*H_1}(P) \geqslant \max\left\{\delta(\epsilon), \frac{C(d)}{\lambda(\epsilon)}\right\}$. However, since β^*H_1 is an ample divisor on X, the set of points

$$E_{X,\phi,\epsilon}(K) := \left\{ P \in I_{X,\phi}(K) \subseteq X(K) \;\middle|\; h_{X,\beta^*H_1}(P) \leqslant \max\left\{\delta(\epsilon), \frac{C(d)}{\lambda(\epsilon)}\right\} \right\} \tag{13}$$

is finite; see [23, Theorem 10.3]. Moreover, for each of the finitely many exceptional points $P \in E_{X,\phi,\epsilon}(K)$, the quantity $\#(\mathcal{O}_{\phi_P}(\beta_P) \cap \mathcal{O}_{K,S})$ is still finite - for wandering points apply (11) and for preperiodic points the entire orbit is finite. Hence,

$$\max_{P\in E_{X,\phi,\epsilon}(K)} \#\big(\mathcal{O}_{\phi_P}(\beta_P) \cap \mathcal{O}_{K,S}\big) < \infty; \tag{14}$$

together with the previous case, this completes the proof of Theorem 1.1. $\square$

Of course, one expects that the actual number of S-integral points in a particular orbit is zero, provided that X has sufficiently many K-rational points. We prove this first in case of constant families over curves of genus at most one, restated from the introduction.

Theorem 1.2. *Let $\varphi \in K(x)$ be such that $\deg(\varphi) \geqslant 2$ and $\varphi^2 \notin \bar{K}[x]$. If X is a curve of genus $g \geqslant 1$ and $\beta : X \to \mathbb{P}^1$ is non-constant, then the set*

$$\{P \in X(K) \mid \big(\mathcal{O}_\varphi(\beta_P) \cap \mathcal{O}_{K,S}\big) \neq \varnothing\}$$

is finite. Moreover, if X is a curve with infinitely many K-rational points (necessarily $g \leqslant 1$), then

$$\overline{\mathrm{Avg}}(\varphi, \beta, S) := \limsup_{B\to\infty} \frac{\sum_{P\in I_{X,\varphi}(B,K)} \#\big(\mathcal{O}_\varphi(\beta_P) \cap \mathcal{O}_{K,S}\big)}{\# I_{X,\varphi}(B,K)} = 0. \tag{5}$$

Note that $I_{X,\phi} = X$ in this case. Before we begin the proof of Theorem 1.2, we need a different sort of bound on the number of integral points in orbits than that given in [8, Corollary 17];

Lemma 2.1. *There exists an $N(\varphi, S) > 0$ such that $\varphi^n(a) \in \mathcal{O}_{K,S}$ implies $n \leqslant N(\varphi, S)$ for all φ-wandering points $a \in \mathbb{P}^1(K)$.*

(Proof of Lemma 2.1). Suppose that $\varphi^n(a) \in \mathcal{O}_{K,S}$ and that $n \geqslant 4$. Since, $\varphi^2 \notin \bar{K}[x]$, it follows from the Riemann-Hurwitz formula that $\#\varphi^{-4}(\infty) \geqslant 3$; see [24, Proposition 3.44]. In particular, the set of S-integral preimages

$$T_4(\varphi, S) := \{b \in \mathbb{P}^1(K) \mid \varphi^4(b) \in \mathcal{O}_{K,S}\} \tag{15}$$

is finite; see [24, Theorem 3.36]. Note that $\varphi^4(\varphi^{n-4}(a)) = \varphi^n(a) \in \mathcal{O}_{K,S}$ and $\varphi^{n-4}(a) \in T_4(\varphi, S)$. Hence, $h(\varphi^{n-4}(a))$ is bounded independently of both a and n. So together with part (2) and (3) of Proposition 2.1, we see that $\deg(\varphi)^{n-4} \cdot \hat{h}_\varphi(a) = \hat{h}_\varphi(\phi^{n-4}(a))$ is bounded. Moreover,

$$\hat{h}_{\varphi,K}^{\min} := \inf\{\hat{h}_\varphi(c) \mid c \in \mathbb{P}^1(K) \text{ wandering for } \varphi\} \tag{16}$$

is strictly positive. To see this, choose an arbitrary wandering point $c_0 \in K$ for φ (possible, for instance, by Northcott's Theorem [24, Theorem. 3.12]), and note that

$$\hat{h}_{\varphi,K}^{\min} = \inf\{\hat{h}_\varphi(c) \mid c \in \mathbb{P}^1(K) \text{ and } 0 < \hat{h}_\varphi(c) < \hat{h}_\varphi(c_0)\}.$$

However, this latter set is finite and consists of strictly positive numbers; hence $\hat{h}_{\varphi,K}^{\min} > 0$. Putting this together with the fact that $\deg(\varphi)^{n-4} \cdot \hat{h}_{\varphi,K}^{\min} \leqslant \deg(\varphi)^{n-4} \cdot \hat{h}_\varphi(a)$ is bounded by the height of points in $T_4(\varphi, S)$, we see that n is bounded independently of a as desired. □

(Proof of Theorem 1.2). Since $\mathrm{PrePer}(\varphi, K)$ is finite [24, Theorem. 3.12] and $\beta : X \to \mathbb{P}^1$ is non-constant, it follows that

$$X_{\varphi,\beta}^{\mathrm{PrePer}}(K) := \{P \in X(K) \mid \beta_P \in \mathrm{PrePer}(\varphi, K)\} \tag{17}$$

is finite. In particular, for both statements of Theorem 1.2, it suffices to assume that $P \in X(K)$ is such that β_P is a wandering point of φ. In light of Lemma 2.1, we define the set

$$T_n(\varphi, \beta, S) := \{P \in X(K) \mid \varphi^n(\beta_P) \in \mathcal{O}_{K,S}\}. \tag{18}$$

Suppose that X has genus $g \geqslant 1$. In this case, it follows from a theorem of Siegel that $T_n(\varphi, \beta, S)$ is finite for all $n \geqslant 0$; see, for instance, [25, Corollary IX 4.3.1]. Moreover, Lemma 2.1, implies that $T_n(\varphi, \beta, S) = \varnothing$ for all $n > N(\varphi, S)$. Hence,

$$\{P \in X(K) \mid \big(\mathcal{O}_\varphi(\beta_P) \cap \mathcal{O}_{K,S}\big) \neq \varnothing\}$$

is finite as claimed.

On the other hand, when X is a rational curve ($g = 0$), we may assume that $X = \mathbb{P}^1$. In this case, $T_n(\varphi, \beta, S)$ can be infinite; see Remark 2.1 below. However, we will show that $T_n(\varphi, \beta, S)$ is sparse in $\mathbb{P}^1(K)$. With this in mind, for any subset $T \subseteq \mathbb{P}^1(K)$, define the upper density $\bar{\delta}_K(T)$ of T to be the quantity

$$\bar{\delta}_K(T) := \limsup_{B\to\infty} \frac{\sum_{\{P\in T \mid H_{\mathbb{P}^1}(P)\leqslant B\}} 1}{\sum_{\{P\in X(K) \mid H_{\mathbb{P}^1}(P)\leqslant B\}} 1}. \tag{19}$$

Note that since $X^{\mathrm{PrePer}}_{\varphi,\beta}(K)$ on (17) is finite, Lemma 2.1 implies that

$$\overline{\mathrm{Avg}}(\varphi, \beta, S) \leqslant N(\varphi, S) \cdot \left(\sum_{n=1}^{N(\varphi,S)} \bar{\delta}_K\big(T_n(\varphi, \beta, S)\big)\right). \tag{20}$$

In particular, it suffices to prove that $\bar{\delta}_K(T_n(\varphi, \beta, S)) = 0$ for all $1 \leqslant n \leqslant N(\varphi, S)$. Letting $f = \varphi^n \circ \beta$, this follows from the following crucial lemma.

Lemma 2.2. *Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be non-constant rational function, and let*

$$T(f, S) := \{b \in \mathbb{P}^1(K) \mid f(b) \in \mathcal{O}_{K.S}\}.$$

Then $\bar{\delta}_K(T(f, S)) = 0$ for all S.

To do this, first note that $T(f, S) \leqslant T(f, S')$ whenever $S \subseteq S'$. Therefore, we may enlarge S and assume that $O_{K,S}$ is a UFD and contains all of f's coefficients.

We prove Lemma 2.2 in cases. Suppose first that f is a polynomial. Again, by enlarging S, we may assume that the leading coefficient of f is an S-unit. Then $f(b) \in \mathcal{O}_{K,S}$ implies that b is integral over $\mathcal{O}_{K,S}$, which is integrally closed. We deduce that $\bar{\delta}_K(T(f, S)) \leqslant \bar{\delta}_K(\mathcal{O}_{K,S}) = 0$ in the polynomial case.

Now suppose that f has a denominator and apply the proof of [24, Theorem 3.36]. In particular, write f in terms of coordinates on $\mathbb{P}^1$:

$$f = [F(x, y), G(x, y)],$$

where $F, G \in \mathcal{O}_{K,S}[x, y]$ are homogenous, coprime polynomials of degree $d = \deg(f) \geqslant 1$. Suppose that $\alpha \in T(f, S)$ and that $\alpha = a/b$ where $a, b \in \mathcal{O}_{K,S}$. Then $G(a, b)$ divides $F(a, b)$ in $\mathcal{O}_{K,S}$, which makes sense, since $O_{K,S}$ is a UFD.

Now, if R is the resultant of F and G, then [24, Proposition 2.13] implies that there are homogeneous polynomials $p_1, q_1, p_2, q_2 \in \mathcal{O}_{K,S}[x, y]$ such that

$$\begin{aligned} p_1(x, y) \cdot F(x, y) + q_1(x, y) \cdot G(x, y) &= R \cdot x^d \\ p_2(x, y) \cdot F(x, y) + q_2(x, y) \cdot G(x, y) &= R \cdot y^d. \end{aligned} \tag{21}$$

Substituting $(x, y) = (a, b)$, we see that $G(a, b)$ must divide R. Hence,

$$T(f, S) \subseteq \bigcup_{r\mid R} \big\{(a, b) \in \mathcal{O}_{K,S} \times \mathcal{O}_{K,S} \bigm| G(a, b) = r\big\}. \tag{22}$$

However, each of the finitely many equations $G(x, y) = r$ cut out an affine variety (possibly reducible) in $\mathbb{A}^2_{/K}$. Let $V(f, S, B)$ be the number of points on the right hand side of (22) of height at most B. It follows from [21, §13.1] that $V(f, S, B) = O\big(B^{[K:\mathbb{Q}]}\big)$. In fact, if each $G(x, y) = r$ is comprised of components of degree at least two, then $V(f, S, B) = O\big(B^{1/2\cdot[K:\mathbb{Q}]} \cdot \log(B)^{\gamma}\big)$ for some $\gamma < 1$; see [21, §13.1 Theorem 2].

On the other hand, there is a constant κ such that the number of points in $\mathbb{P}^1(K)$ of height at most B is asymptotic to $\kappa \cdot B^{2[K:\mathbb{Q}]}$; this is a special case of a theorem of Schanuel [19]. In any case, the density $\overline{\delta}_K(T(f,S)) = 0$ as claimed. □

Remark 2.1. When $g = 0$, it is possible that $T_n(\varphi, \beta, S)$ is infinite, even if one assumes $\varphi^2 \notin K[x]$. For example, let $F(x) \in \mathbb{Z}[x]$ be any polynomial of degree $2d$, let $D > 1$ be any square-free integer, and let

$$\varphi(x) = \frac{F(x)}{(x^2 - D)^d}. \tag{23}$$

If $(u, v) \in \mathbb{Z}^2$ is a solution to the Pell equation $u^2 - Dv^2 = 1$, then $\varphi(u/v) = v^{2d} \cdot F(u/v) \in \mathbb{Z}$. Moreover, there are infinitely many such solutions. Setting $\beta(t) = t$, we see that $T_n(\varphi, \beta, S)$ is infinite. However, the set of rational numbers u/v satisfying $u^2 - Dv^2 = 1$ are sparse in $\mathbb{P}^1(\mathbb{Q})$.

We would like to extend Theorem 1.2 to non-constant families of rational maps. To do this, note that if (X, ϕ, β) is a family as in Theorem 1.1, then the average number of S-integral points in $\mathcal{O}_{\phi_P}(\beta_P)$ is bounded. In particular, such families are a good place to test generalizations of Theorem 1.2. In order to distill the additional properties needed, we study the following family:

Theorem 2.1. *Let $\phi : \mathbb{P}^1 \to \mathrm{Rat}_3$ be the family of rational functions given by*

$$\phi_t(x) := \frac{x - t}{x^3 + 1}, \quad \textit{for all} \;\; t \in \mathbb{P}^1(\mathbb{Q}) \setminus \{-1, \infty\}. \tag{24}$$

If $\beta \in \mathbb{Q}(t)$ satisfies $\deg(\beta) \geqslant 3$, then $\overline{\mathrm{Avg}}(\phi, \beta, \mathbb{Z}) = 0$.

Proof. First, we compute the second iterate

$$\phi_t^2(x) := \frac{f_t(x)}{g_t(x)} = \frac{-tx^9 + x^7 - 4tx^6 + 2x^4 - 5tx^3 + x - 2t}{x^9 + 3x^6 + 4x^3 - 3tx^2 + 3t^2x - t^3 + 1} \tag{25}$$

and the resultant $\mathrm{Res}_x(f_t, g_t) = (t+1)^{12} \cdot (t^2 - t + 1)^{12}$, and deduce that if $t \neq -1$, then $\phi_t^2 \notin \bar{\mathbb{Q}}[x]$. Hence, $I_{\mathbb{P}^1,\phi}(\mathbb{Q}) = \mathbb{P}^1(\mathbb{Q}) \setminus \{-1, \infty\}$; see (2). From here, we show the existence of $N(\phi, \beta)$, a uniform largest iterate producing an integer point ; compare to Lemma 2.1 above.

Write $t = a/b$ for some coprime $a, b \in \mathbb{Z}$ and suppose that $\phi_t^n(\beta_t) = \phi_t(\phi_t^{n-1}(\beta_t)) \in \mathbb{Z}$. On the other hand, if we write $\phi_t([x, y]) = \big[bxy^2 - ay^3, b(x^3 + y^3)\big]$ in terms of coordinates on $\mathbb{P}^1$, then the proof of [24, Thoerem 3.36] (which applies since $t \neq 1$) implies that

$$\left\{ \frac{x}{y} \in \mathbb{Q} \;\middle|\; \phi_t\Big(\frac{x}{y}\Big) \in \mathbb{Z} \right\} \subseteq \bigcup_{r \mid (a^3 + b^3)} \big\{x, y \in \mathbb{Z} \mid x^3 + y^3 = r\big\}.$$

Note that $|r| \leqslant 2H(t)^3$, and combined with Lemma 2.3 below, we see that

$$H(x/y) \leqslant 2\sqrt{2} \cdot H(t)^{\frac{3}{2}}, \quad \text{whenever } \; \phi_t\left(\frac{x}{y}\right) \in \mathbb{Z}. \tag{26}$$

In particular, $h(\phi_t^{n-1}(\beta_t)) \leqslant 3/2 \cdot h(t) + \log(2\sqrt{2})$. Moreover, parts (2) and (3) of Proposition 2.1 imply that

$$\begin{aligned} 3^{n-1} \cdot \hat{h}_{\phi_t}(\beta_t) = \hat{h}_{\phi_t}(\phi_t^{n-1}(\beta_t)) &\leqslant h(\phi_t^{n-1}(\beta_t)) + 5/2 \cdot h(\phi_t) + c_3/2 \\ &\leqslant 3/2 \cdot h(t) + \log(2\sqrt{2}) + 5/2 \cdot h(\phi_t) + c_3/2 \end{aligned} \tag{27}$$

On the other hand, $h(\phi_t) = h([1, -t, 1, 1]) = h(t)$, so that (27) implies that

$$3^{n-1} \cdot \hat{h}_{\phi_t}(\beta_t) \leqslant 4 \cdot h(t) + \log(2\sqrt{2}) + c_3/2.$$

Finally, parts (1) and (2) of Proposition 2.1 give the lower bound

$$(\deg(\beta) - 5/2)\cdot h(t) - B_1 - c_3/2 \leqslant \hat{h}_{\phi_t}(\beta_t). \tag{28}$$

However, $\deg(\beta) \geqslant 3 > 5/2$ by assumption, and we deduce that

$$3^{n-1} \leqslant \frac{4\cdot h(t) + \log(2\sqrt{2}) + c_3/2}{(\deg(\beta) - 5/2)\cdot h(t) - B_1 - c_3/2},$$

for all but finitely many $t \in \mathbb{Q}$. Hence, $\phi_t^n(\beta_t) \in \mathbb{Z}$ implies that n is bounded for such t. On the other hand, since $(\mathcal{O}_{\phi_t}(\beta_t) \cap \mathbb{Z})$ is finite for all $t \neq -1$, the quantity $\max\{n \mid \phi_t^n(\beta_t) \in \mathbb{Z}, t \in T\}$ is bounded for any finite subset $T \subseteq \mathbb{Q} \setminus \{-1\}$. We conclude that there is an integer $N(\phi,\beta)$, such that $\phi_t^n(\beta_t) \in \mathbb{Z}$ implies $n \leqslant N(\phi,\beta)$ for all $t \neq -1$; compare to Lemma 2.1.

As in the proof of Theorem 1.2, the density estimate in Lemma 2.2 implies $\overline{\delta}_{\mathbb{Q}}\big(T(\phi^n\circ\beta,\mathbb{Z})\big) = 0$ for all $n \leqslant N(\phi,\beta)$. It follows that $\overline{\mathrm{Avg}}(\phi,\beta,\mathbb{Z}) = 0$ as claimed. □

Lemma 2.3. *Suppose that $x^3 + y^3 = B$, for some integers $x, y, B \in \mathbb{Z}$ with $B \neq 0$. Then*

$$\max\big\{|x|,|y|\big\} \leqslant 2\sqrt{|B|}. \tag{29}$$

(Proof of Lemma 2.3). We factor $x^3 + y^3$ in $\mathbb{Q}[x,y]$, and write

$$B = x^3 + y^3 = (x+y)\cdot(x^2 - xy + y^2) = (x+y)\cdot\bigg(\frac{3}{4}(x-y)^2 + \frac{1}{4}(x+y)^2\bigg).$$

In particular, we see that

$$\max\Big\{3/4\cdot(x-y)^2,\ 1/4\cdot(x+y)^2\Big\} \leqslant |B|, \tag{30}$$

since $|x+y| \geqslant 1$ and both terms on the left side of (30) are positive. On the other hand, it is straightforward to verify that

$$\max\big\{|x|,|y|\big\} \leqslant \max\Big\{|x-y|, |x+y|\Big\}, \tag{31}$$

and we deduce from (30) and (31) that $\max\{|x|,|y|\} \leqslant 2\sqrt{|B|}$ as claimed. □

Remark 2.2. The argument in Theorem 2.1 applies to any family $\phi_t(x) = p(t,x)/(x^3+1)$, where $p \in \mathbb{Z}[t,x]$ and $\deg_x(p) \leqslant 3$. The key point is that the linear height bound in Lemma 2.3 remains unchanged. We discuss the implications of such height bounds in general in the following section.

Height uniformity conjectures and averages in families

We would like to extend Theorem 1.2 to non-constant families of rational maps $\phi : X \to \mathrm{Rat}_d$ parametrized by varieties of arbitrary dimension. To do this, we must translate the strategy of the proofs of Theorems 1.2 and 2.1 to more general language. In particular, our goal (loosely speaking) is to show the following property:

$$\boxed{\Big\{P \in I_{X,\phi}(K)\ :\ \big(\mathcal{O}_{\phi_P}(\beta_P) \cap \mathcal{O}_{K,S}\big) \neq \varnothing\Big\} \subseteq X \text{ is thin,}} \tag{32}$$

for all sufficiently generic basepoint families $\beta : X \to \mathbb{P}^1$; see [21, §3.1] for the definition of thin. From here, if X has sufficiently many rational points and the proper subvariety (when $\dim(X) \geqslant 2$) containing (32) is small enough, then one expects that $\overline{\mathrm{Avg}}(\phi,b,S) = 0$.

The main technique we used to establish (32) for families over curves was the existence of a uniform largest iterate $N(\phi,\beta,S)$ that could produce an S-integral point; see Lemma 2.1. To find such an $N(\phi,\beta,S)$, we used the fact there was a linear lower bound (in terms of a height

function on X) on both the canonical height $\hat{h}_{\phi_P}(\beta_P)$ and the height of points $Q \in \mathbb{P}^1(K)$ satisfying $\phi_P^4(Q) \in \mathcal{O}_{K,S}$; see (10) and Lemma 2.3.

For curves X and families (X, ϕ, β) as in Theorem 1.1, the lower bound on the canonical height $\hat{h}_{\phi_P}(\beta_P)$ is not a problem; see (10) in the proof of Theorem 1.1. In general, it is predicted by a dynamical analog [24, Conjecture 4.98] of a conjecture of Lang [14, page 92] regarding the canonical height of a non-torsion point on an elliptic curve. Likewise, the lower bound on the height of points $Q \in \mathbb{P}^1(K)$ such that $\phi_P^4(Q) \in \mathcal{O}_{K,S}$ follows from several height-uniformity conjectures in arithmetic geometry, including the following:

Conjecture 3.1. *Let $\mathcal{C} \to \mathcal{B}$ be a family of positive genus curves equipped with a map $f \in K(\mathcal{C})$ that is non-constant on fibers. Then there are constants κ_1 and κ_2 such that*

$$h_{\mathcal{C}}(Q) \leqslant \kappa_1 \cdot h_{\mathcal{B}}(P) + \kappa_2 \qquad \textit{for all} \quad \{Q \in \mathcal{C}_P(K) \mid f_P(Q) \in \mathcal{O}_{K,S}\}, \tag{33}$$

whenever the fiber $\mathcal{C}_P$ is smooth; here, $h_{\mathcal{C}}$ is an arbitrary height function and h_β is ample.

Remark 3.1. For elliptic curves, versions of Conjecture 3.1 were made by Hall and Lang [25, IV.7]. Moreover, Conjecture 3.1 is a consequence of the Vojta conjecture [31, §3.4.3]; for justification, see [11, Theorem 1.0.1] for the case of elliptic curves and [10] or [29, Conjecture 4] for the case of higher genus. Over function fields, linear height bounds such as those on (33) have appeared in [12, 16, 20, 30].

Assuming Conjecture 3.1, we prove an averaging result analogous to Theorems 1.2 and 2.1.

Theorem 3.1. *Let $X_{/K}$ be a curve of genus g, and suppose that $\phi : X \to \mathrm{Rat}_d \subseteq \mathbb{P}^{2d+1}$ and $\beta : X \to \mathbb{P}^1$ are rational maps that satisfy*

$$\deg(\beta) > \frac{2d-1}{d-1} \cdot \deg(\phi^* H)$$

for the hyperplane class $H \in \mathrm{Pic}(\mathbb{P}^{2d+1})$. Then Conjecture 3.1 implies that the set

$$\{P \in I_{X,\phi}(K) \mid \big(\mathcal{O}_{\phi_P}(\beta_P) \cap \mathcal{O}_{K,S}\big) \neq \varnothing\}$$

is finite whenever $g \geqslant 1$. Moreover, if X is a curve with infinitely many K-rational points, then write $\phi^2 = f/g$ for some $f, g \in K(X)[z]$. If $\deg(g) \geqslant 1$ and the resultant $\mathrm{Res}(f,g) \neq 0$, then

$$\overline{\mathrm{Avg}}(\phi, \beta, S) := \limsup_{B \to \infty} \frac{\sum_{P \in I_{X,\phi}(B,K)} \#\big(\mathcal{O}_{\phi_P}(\beta_P) \cap \mathcal{O}_{K,S}\big)}{\# I_{X,\phi}(B,K)} = 0. \tag{34}$$

Proof. Since there is no harm in enlarging S, we may assume that the ring $\mathcal{O}_{K,S}$ is a UFD. For $P \in I_{X,\phi}(K)$, write $\phi_P^4 = [F_{(P,4)}, G_{(P,4)}]$ for some coprime, homogeneous polynomials $F_{(P,4)}, G_{(P,4)} \in \mathcal{O}_{K,S}[x,y]$ of degree $\deg(\phi)^4$. As in the proof of [24, Thoerem 3.36], if $Q = [a,b] \in P^1(K)$ is such that $\phi_P^4(Q) \in \mathcal{O}_{K,S}$, then $G_{(P,4)}(a,b)$ divides the resultant $R_{(P,4)} := \mathrm{Res}(F_{(P,4)}, G_{(P,4)})$; here we assume that $a, b \in \mathcal{O}_{K,S}$.

In particular, if we consider the family of curves $\mathcal{C} \to X \times \mathbb{P}^1$ with fibers $\mathcal{C}_{(P,r)} : G_{(P,4)}(x,y) = r$ and non-constant maps $x, y : \mathcal{C}_{(P,r)} \to \mathbb{P}^1$, then it follows from Conjecture 3.1 that there exist absolute constants $\kappa_{1,x}, \kappa_{2,x}, \kappa_{1,y}, \kappa_{2,y} > 0$ satisfying

$$h_{\mathbb{P}^1}(x(Q)) \leqslant \kappa_{1,x} \cdot \big(h_X(P) + h_{\mathbb{P}^1}(R_{(P,4)})\big) + \kappa_{2,x} \quad \text{and} \quad h_{\mathbb{P}^1}(y(Q)) \leqslant \kappa_{1,y} \cdot \big(h_X(P) + h_{\mathbb{P}^1}(R_{(P,4)})\big) + \kappa_{2,y};$$

here we take the ample height $h_{X \times \mathbb{P}^1} = h_X + h_{\mathbb{P}}^1$; see [24, Example 7.28]. On the other hand, since the map $P \to R_{(P,4)}$ determines a morphism $X \to \mathbb{P}^1$ and all heights are dominated by ample

heights [15, Chap. 4, Prop 5.4], we see that $h_{\mathbb{P}^1}(R_{(P,4)}) \leqslant c_1 \cdot h_X(P) + c_2$ for some constants $c_1, c_2 > 0$. Combined with the bounds above, we deduce that

$$h_{\mathbb{P}^1}(Q) \leqslant \kappa_1 \cdot h_X(P) + \kappa_2, \tag{35}$$

for some constants $\kappa_1, \kappa_2 > 0$. Now, if $P \in I_{X,\phi}(K)$ and $\phi_P^n(\beta_P) \in \mathcal{O}_{K,S}$ for some $n \geqslant 4$, then $\phi_P^4(\phi_P^{n-4}(\beta_P)) \in \mathcal{O}_{K,S}$. Letting $Q = \phi_P^{n-4}(\beta_P)$, we see that (35) and Proposition 2.1 parts (1) and (2) imply that

$$d^{n-4} \cdot \hat{h}_{\phi_P}(\beta_P) \leqslant \kappa_1' \cdot h_X(P) + \kappa_2'. \tag{36}$$

Finally, since $\deg(\beta) > \frac{2d-1}{d-1} \cdot \deg(\phi^* H)$ for the hyperplane class $H \in \operatorname{Pic}(\mathbb{P}^{2d+1})$, the lower bound on (10) and (36) imply that

$$d^{n-4} \leqslant \frac{a \cdot h_X(P) + b}{c \cdot h_X(P) + d},$$

for all but finitely many $P \in I_{X,\phi}(K)$; here $a, c > 0$ are positive constants and b, d are additional constants. In particular, there exists $N(\phi, \beta, S)$ such that $\phi_P^n(\beta_P) \in \mathcal{O}_{K,S}$ implies $n \leqslant N(\phi, \beta, S)$ for all but finitely many $P \in I_{X,\phi}(K)$. As in the proof of Theorem 1.2, when the genus of X is positive, the first statement in Theorem 3.1 follows from Siegel's Theorem [25, Corollary IX 4.3.1] applied to the rational maps $f_n : X \to \mathbb{P}^1$ defined by $f_n(P) = \phi_P^n(\beta_P)$ for all $0 \leqslant n \leqslant N(\phi, \beta, S)$.

On the other hand, suppose that X has infinitely many K-rational points. We may view ϕ as an element of $K(X)(z)$, and write $\phi^2 = f/g$ for some polynomials $f, g \in K(X)[z]$. By assumption, $\deg_z(g) \geqslant 1$ and the resultant $\operatorname{Res}_{K(X)}(f, g) \neq 0$; here $\operatorname{Res}_{K(X)}(f, g)$ is an element of $K(X)$. Since, evaluation at P gives a ring homomorphism $K(X) \to K$, it follows from the definition of the resultant as a Vandermonde determinant [13, IV.§8], that $\operatorname{Res}_K(f_P, g_P) = \operatorname{Res}_{K(X)}(f, g)(P)$, as elements of K, whenever $\deg(f) = \deg(f_P)$ and $\deg(g) = \deg(g_P)$. Hence, there are only finitely many $P \in X(\bar{K})$ such that $\operatorname{Res}_K(f_P, g_P) = 0$. But if $P \in X(K) \setminus I_{(X,\phi)}(K)$, then $\deg(g_P) < 1$ or $\operatorname{Res}_K(f_P, g_P) = 0$. Therefore, $X(K)$ and $I_{(X,\phi)}(K)$ have the same number of points asymptotically. The rest of Theorem 3.1 follows directly from Lemma 2.2. $\square$

Because of it's independent interest, we state the following corollary of the proof of Theorem 3.1, establishing the existence of uniform largest iterate that can produce an S-integral point.

Corollary 3.1. *Let (X, ϕ, β) be as in Theorem 3.1. Then Conjecture 3.1 implies that*

$$\left\{n \mid P \in I_{X,\phi}(K),\ \phi_P^n(\beta_P) \in \mathcal{O}_{K,S},\ \textit{and}\ \hat{h}_{\phi_P}(\beta_P) \neq 0\right\} \tag{37}$$

is bounded.

It is clear that the key fact in our study of integral points in orbits in families $\phi : X \to \operatorname{Rat}_d$ is the lower bound on (10). Unfortunately, such a bound is unlikely to hold in full generality when X has dimension at least two, since the set of points $P \in X(\bar{K})$ such that $\hat{h}_{\phi_P}(\beta_P) = 0$ is large: it contains the codimension one subvariety of points satisfying $\phi_P(\beta_P) = \beta_P$.

Remark 3.2. Though canonical heights in higher dimensional dynamical systems are not well understood, there has been progress made on bounding the average value canonical heights in higher dimensional families of elliptic curves; see [32, 33]. It would be interesting to know if (or when) such techniques generalize, for instance to morphisms of $\mathbb{P}^n$.

On the other hand, if (X, ϕ, β) satisfy the degree conditions of Theorem 1.1, a bound such as (10) likely holds on a nontrivial, proper open subset $U \subseteq X$. Therefore, if we can control the size of $\#(\mathcal{O}_{\phi_P}(\beta_P) \cap \mathcal{O}_{K,S})$ on the complement of U, we can bound $\overline{\operatorname{Avg}}(\phi, \beta, S)$. We illustrate this

idea with the following 3-dimensional example; however, for reasons that will become clear in the proof, we must restrict ourselves to integer specializations.

Theorem 3.2. *Let $\phi : \mathbb{A}^1 \times \mathbb{A}^1 \times \mathbb{A}^1 \to \mathrm{Rat}_3$ be the family of rational maps:*

$$\phi_{(r,s,t)}(x) := \frac{(r \cdot s) \cdot x^3 + s \cdot x + t}{x^2+1} \quad \text{for } r, s, t \in \mathbb{Z}. \tag{38}$$

If $\beta : \mathbb{A}^1 \times \mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{A}^1$ is the map $\beta_{(r,s,t)} = r^{n_1} \cdot s^{n_2} \cdot t^{n_3}$ with $\min\{n_1, n_2, n_3\} \geqslant 6$, then

$$\overline{\mathrm{Avg}}_{\mathbb{Z}}(\phi, \beta, \mathbb{Z}) := \limsup_{B\to\infty} \frac{\sum_{|r|,|s|,|t|\leqslant B} \#\Big(\mathcal{O}_{\phi_{(r,s,t)}}\big(\beta_{(r,s,t)}\big) \cap \mathbb{Z}\Big)}{(2B+1)^3} \tag{39}$$

is bounded.

A priori, it is not clear that $\mathcal{O}_{\phi_{(r,s,t)}}\big(\beta_{(r,s,t)}\big)$ contains only finitely many integers for all $r, s, t \in \mathbb{Z}$; for instance, $\phi_{(1,s,0)}(x) = sx$ is a polynomial, and [24, Theorem 3.43] does not apply. However, the basepoint $\beta_{(1,s,0)} = 0$ is fixed in this case, and so finiteness is not a problem. Before we begin the proof of Theorem 3.2, we prove the stronger statement: that $\mathcal{O}_{\phi_{(r,s,t)}}\big(\beta_{(r,s,t)}\big)$ contains only finitely many integers for all rational values of $r, s, t \in \mathbb{Q}$, not just integer values as in Theorem 3.2.

Lemma 3.1. *The orbit $\mathcal{O}_{\phi_{(r,s,t)}}\big(\beta_{(r,s,t)}\big)$ contains only finitely many integers for all $r, s, t \in \mathbb{Q}$.*

(Proof of Lemma 3.1). Note that if $t = 0$, then $\beta_{(r,s,t)} = 0$ and $\phi_{(r,s,t)}(0) = 0$. In particular, $\mathcal{O}_{\phi_{(r,s,t)}}\big(\beta_{(r,s,t)}\big) = \{0\}$, and there is nothing left to prove in this case. Therefore, without loss of generality, we may assume that $t \neq 0$. Likewise, if $r \cdot s = 0$, then $\phi_{(r,s,t)}$ is a bounded function on the real line. In particular, the set of integers of $\mathcal{O}_{\phi_{(r,s,t)}}\big(\beta_{(r,s,t)}\big)$ is a finite.

In the remaining cases, it suffices to show that $\phi^2_{(r,s,t)} \notin \mathbb{Q}[x]$; see [24, Theorem 3.43]. We compute that

$$\phi^2_{(r,s,t)}(x) := \frac{f_{(r,s,t)}(x)}{g_{(r,s,t)}(x)} := \frac{(r^4s^4)x^9 + (3r^3s^4 + rs^2)x^7 + (3r^3s^3t + t)x^6 + \cdots + (rst^3 + st + t)}{(r^2s^2)x^8 + (r^2s^2 + 2rs^2 + 1)x^6 + (2rst)x^5 + \cdots + (t^2+1)}.$$

Therefore, if $\phi^2_{(r,s,t)} \in \mathbb{Q}[x]$, then $g_{(r,s,t)}(x) \cdot (ax + b) = f_{(r,s,t)}(x)$ for some $a, b \in \mathbb{Q}$. By equating the x^9 coefficient, we see that $a := (st)^2$. Moreover, after substituting $a := (st)^2$ and examining the x^8 coefficient, we see that $b = 0$. Similarly, since, $b = 0$ and $t \neq 0$, the x^6 coefficient implies that $(rs)^3 = -1$, and we deduce that $(rs) = -1$. In particular, the x^7 coefficient implies that $s = -1$. Finally, after substituting $s = -1$ in the constant term expression, we see that $t = 0$, a contradiction. □

(Proof of Theorem 3.2). We begin by establishing a lower bound on the canonical height $\hat{h}_{\phi_{(r,s,t)}}(\beta_{(r,s,t)})$ as on (10) for all $r, s, t \in \mathbb{Z}$ in an open subset of $X := \mathbb{A}^1 \times \mathbb{A}^1 \times \mathbb{A}^1$. In particular, we see that if $r, s, t \in \mathbb{Z}$ are such that $r \cdot s \cdot t \neq 0$, then

$$h(\beta_{(r,s,t)}) = h(r^{n_1} \cdot s^{n_2} \cdot t^{n_3}) = \log\big(|r^{n_1}| \cdot |s^{n_2}| \cdot |t^{n_3}|\big) \geqslant \min_{1\leqslant i\leqslant 3}\{n_i\} \cdot \log\max\big\{|r|, |s|, |t|\big\}. \tag{40}$$

Let $U :=\subseteq \mathbb{A}^1 \times \mathbb{A}^1 \times \mathbb{A}^1$ be the open subset of points $P := (r, s, t)$ such that $r \cdot s \cdot t \neq 0$, and define the height function $h_X(P) = \log\max\big\{|r|, |s|, |t|\big\}$. Then, (40) implies that

$$h(\beta_P) \geqslant \min_{1\leqslant i\leqslant 3}\{n_i\} \cdot h_X(P), \quad \text{for all } P \in U(\mathbb{Z}); \tag{41}$$

here $U(\mathbb{Z})$ denotes the set of points of U with integral coordinates. Note that the bound on (41) will not hold on $U(\mathbb{Q})$ in general.

On the other hand, it is easy to see that

$$h(\phi_P) := h_{\mathbb{P}^7}\big([r\cdot s, 0, s, t, 0, 1, 0, 1]\big) \leqslant 2\cdot \log\max\big\{|r|, |s|, |t|\big\} = 2\cdot h_X(P), \tag{42}$$

for all $P \in X(\mathbb{Z})$. In particular, part (2) of Proposition 2.1 and the bounds on (41) and (42) yield

$$\hat{h}_{\phi_P}(\beta_P) \geqslant \big(\min_{1\leqslant i\leqslant 3}\{n_i\} - 5\big)\cdot h_X(P) - C, \quad \text{for all } P \in U(\mathbb{Z}); \tag{43}$$

here C is an absolute, positive constant. Hence, as in the proofs of all previous theorems, the bound in [8, Corollary 17] implies that $\#(\mathcal{O}_{\phi_P}(\beta_P)\cap\mathbb{Z}) \leqslant M(\phi,\beta)$ is bounded uniformly over all points $P \in U(\mathbb{Z})$; note that we have used crucially the fact that $\min\{n_1, n_2, n_3\} \geqslant 6$. It remains to control $\#(\mathcal{O}_{\phi_P}(\beta_P)\cap\mathbb{Z})$ for all $P = (r, s, t)$ such that $r\cdot s\cdot t = 0$ on average. We do this in cases.

Suppose first that $t = 0$. Then one computes that $\phi_{(r,s,0)}(\beta_{(r,s,0)}) = \phi_{(r,s,0)}(0) = 0$; hence

$$\#\big(\mathcal{O}_{\phi_{(r,s,0)}}(\beta_{(r,s,0)})\cap\mathbb{Z}\big) = 1, \quad \text{for all } r, s \in \mathbb{Z}. \tag{44}$$

On the other hand, if $s = 0$, then $\phi_{(r,0,t)}(x) = t/(x^2+1)$. Therefore, $|\phi_{(r,0,t)}(x)| \leqslant |t|$ is a bounded function on the real line. We deduce that,

$$\#\big(\mathcal{O}_{\phi_{(r,0,t)}}(\beta_{(r,0,t)})\cap\mathbb{Z}\big) \leqslant 2B+1, \quad \text{when } \max\{|r|, |t|\} \leqslant B. \tag{45}$$

Finally, suppose that $r = 0$. Then $\phi_{(0,s,t)}(x) = (sx+t)/(x^2+1)$ is also a bounded function on the real line. Specifically, $|\phi_{(0,s,t)}(x)| \leqslant |s| + |t|$ follows from elementary calculus. Therefore,

$$\#\big(\mathcal{O}_{\phi_{(0,s,t)}}(\beta_{(0,s,t)})\cap\mathbb{Z}\big) \leqslant 4B+1, \quad \text{when } \max\{|s|, |t|\} \leqslant B. \tag{46}$$

We deduce from (44), (45), and (46) that

$$\begin{aligned}
\frac{\sum_{|r|,|s|,|t|\leqslant B}\#\Big(\mathcal{O}_{\phi_{(r,s,t)}}\big(\beta_{(r,s,t)}\big)\cap\mathbb{Z}\Big)}{(2B+1)^3} &= \frac{\sum_{P\in U(\mathbb{Z},B)}\#\Big(\mathcal{O}_{\phi_P}\big(\beta_P\big)\cap\mathbb{Z}\Big)}{(2B+1)^3} + \frac{\sum_{|s|,|t|\leqslant B}\#\Big(\mathcal{O}_{\phi_{(0,s,t)}}\big(\beta_{(0,s,t)}\big)\cap\mathbb{Z}\Big)}{(2B+1)^3}\\
&\quad + \frac{\sum_{|r|,|t|\leqslant B}\#\Big(\mathcal{O}_{\phi_{(r,0,t)}}\big(\beta_{(r,0,t)}\big)\cap\mathbb{Z}\Big)}{(2B+1)^3} + \frac{\sum_{|r|,|s|\leqslant B}\#\Big(\mathcal{O}_{\phi_{(r,s,0)}}\big(\beta_{(r,s,0)}\big)\cap\mathbb{Z}\Big)}{(2B+1)^3}\\
&\leqslant \frac{\sum_{P\in U(\mathbb{Z},B)} M(\phi,\beta)}{(2B+1)^3} + \frac{\sum_{|s|,|t|\leqslant B}(4B+1)}{(2B+1)^3} + \frac{\sum_{|r|,|t|\leqslant B}(2B+1)}{(2B+1)^3} + \frac{\sum_{|r|,|s|\leqslant B}1}{(2B+1)^3}
\end{aligned}$$

Letting B tend to infinity, we see that

$$\overline{\mathrm{Avg}}_{\mathbb{Z}}(\phi,\beta,\mathbb{Z}) \leqslant M(\phi,\beta) + 3, \tag{47}$$

and the average number of integral points is bounded as claimed; here, $M(\phi,\beta)$ is the bound on $\#(\mathcal{O}_{\phi_P}\big(\beta_P\big)\cap\mathbb{Z})$ for all $P \in U(\mathbb{Z})$ obtained from (43) and [8, Corollary 17]. $\square$

Integral points in orbits over function fields and higher dimensions

Since the techniques in this paper translate well to global function fields $K/\mathbb{F}_q(t)$, it is likely that many of our results hold in this setting, once one establishes a suitable version of Silverman's finiteness theorem; see, for instance, [9]. In fact, there is even more hope of proving the zero-average statement in Theorem 3.1 unconditionally over function fields, since the linear height bound in Conjecture 3.1 is known; see [12, 30].

Remark 4.1. Interestingly, the are many additional applications of Conjecture 3.1 to arithmetic dynamics, including to problems in dynamical Galois theory, i.e. the study of Galois groups of iterates of rational functions, and to prime factorization problems in orbits; see [4, 5, 6].

This is not to say that the function field case comes without added complexity. Namely, one must often assume that $\varphi \in K(x)$ is non-isotrivial, meaning that we exclude functions that are defined (after a change of variables) over the field of constants of K; see [9, Definition 1]. Furthermore, it is not immediately obvious what type of subset (or subvariety) the set of isotrivial rationals maps IsoRat_d are inside Rat_d. More pertinently, if $X_{/K}$ is a curve, can we even construct families $\phi : X \to \mathrm{Rat}_d$ whose images are non-isotrivial (and separable)? As motivation for future research, we we prove that $\overline{\mathrm{Avg}}(\phi, \beta, S) = 0$ for all non-isotrivial families $\phi : X \to \mathrm{Rat}_d$ acting periodically at infinity; see Theorem 4.1 below. To do this, we fix some notation.

Let $K/\mathbb{F}_q(t)$ be a finite extension corresponding to a curve $C_{/\mathbb{F}_q}$ with $K = \mathbb{F}_q(C)$, and let $\mathcal{O}_K$ be the integral closure of $\mathbb{F}_q[t]$ in K. For all finite sets S of primes ideals $\mathfrak{q} \subseteq \mathcal{O}_K$, we form the ring of S-integers as in the number field case; see [18] and [28] for introductions to the arithmetic of function fields. The prime ideals of $\mathcal{O}_K$ correspond to a complete set of absolute values (satisfying the product formula) M_K on K. In particular, we may define the standard Weil height function $h_K : \mathbb{P}^1(K) \to \mathbb{R}_{\geqslant 0}$ for $P = [z_0, z_1] \in \mathbb{P}^1(K)$ by

$$h_K(P) := \sum_{v \in M_K} \log\max\{|z_0|_v, |z_1|_v\}; \tag{48}$$

note that all absolute values on K are non-archimedian, a key distinction with that of number fields.

Since Silverman's finiteness theorem is not known in full generality over over function fields, we need some additional assumptions. To wit, for rational functions $\varphi \in K(x)$, define the set

$$\mathrm{Per}_m(\varphi) := \{P \in \mathbb{P}^1(\bar{K}) \mid \varphi^m(P) = P\}, \tag{49}$$

of points fixed by some iterate of φ. Finally, since iterating inseparable maps ($\varphi' = 0$) over fields of finite characteristic can add difficulty [27], we define the separable degree $d_s(\varphi)$ of φ to be the separable degree of the extension $K(x)/K(\varphi)$. With these notions in place, we our ready to define the set of good specializations (c.f (2)) in a family $\phi : X \to \mathrm{Rat}_d$ over K:

(50)

$$I_{X,\phi,m} := \big\{P \in X \;\big|\; \phi_P \text{ is defined},\ \phi_P \notin \mathrm{IsoRat}_d,\ \phi_P^2 \notin \bar{K}[x],\ d_s(\phi_P) \geqslant 2,\ \infty \in \mathrm{Per}_m(\phi_P)\big\}.$$

In particular, $(\mathcal{O}_{\phi_P}(b) \cap \mathcal{O}_{K,S})$ is finite for all $b \in K$ and $P \in I_{X,\phi,m}(K)$ by [9, Theorem 2(ii)].

Remark 4.2. Strictly speaking, the authors of [9] do not state this conclusion, since there they consider only function fields with algebraically closed ground fields, where the Northcott property fails. However, the finiteness of $(\mathcal{O}_{\phi_P}(b) \cap \mathcal{O}_{K,S})$ for $b \in K$ and $P \in I_{X,\phi,m}(K)$ follows from the height bound in [9, Theorem 2(ii)], followed by the existence of a positive lower bound $\hat{h}^{\min}_{\phi_P,K}$ on the canonical height $\hat{h}_{\phi_P}(b)$ over all wandering points $b \in \mathbb{P}^1(K)$; compare to (16) above.

Theorem 4.1. *Let $K/\mathbb{F}_q(t)$, let $X_{/K}$ be a curve, and let $d \geqslant 2$. If $\phi : X \to \mathrm{Rat}_d$ and $\beta : X \to \mathbb{P}^1$ are separable rational maps and*

$$\deg(\beta) > \frac{2d-1}{d-1} \cdot \deg(\phi^* H), \tag{51}$$

for the hyperplane class $H \in \mathrm{Pic}(\mathbb{P}^{2d+1})$, then

$$\big\{n \mid P \in I_{X,\phi,m}(K),\ \phi_P^n(\beta_P) \in \mathcal{O}_{K,S},\ \textit{and}\ \hat{h}_{\phi_P}(\beta_P) \neq 0\big\} \tag{52}$$

is bounded. In particular, it follows that $\#\big(\mathcal{O}_{\phi_P}(\beta_P)\cap\mathcal{O}_{K,S}\big)$ *is uniformly bounded over all points* $P\in I_{X,\phi,m}(K)$*. Moreover, if* $X=\mathbb{P}^1$ *and* $X(K)\setminus I_{X,\phi,m}(K)$ *is finite, then*

$$\overline{\operatorname{Avg}}(\phi,\beta,S):=\limsup_{B\to\infty}\frac{\sum_{P\in I_{X,\phi,m}(B,K)}\#\big(\mathcal{O}_{\phi_P}(\beta_P)\cap\mathcal{O}_{K,S}\big)}{\#I_{X,\phi,m}(B,K)}=0 \tag{53}$$

Remark 4.3. Note that Theorem 4.1 implies that $\overline{\operatorname{Avg}}(\phi,\beta,S)=0$ for all suitable constant families $\phi:\mathbb{P}^1\to\operatorname{Rat}_d$ and all non-constant $\beta:\mathbb{P}^1\to\mathbb{P}^1$; hence, we recover Theorem 1.2 in this setting.

In particular, we may apply Theorem 4.1 to the following explicit non-constant family.

Corollary 4.1. *Let* $K/\mathbb{F}_q(t)$*, let* $d\geqslant 2$*, and let* $\phi:\mathbb{P}^1\to\operatorname{Rat}_d$ *be the rational map given by*

$$\phi(f)(x)=\frac{(f+1)\cdot x^d}{x^{d-1}+f},\qquad \textit{for } f\in K. \tag{54}$$

Then for all separable maps $\beta:\mathbb{P}^1\to\mathbb{P}^1$ *satisfying* $\deg(\beta)>(2d-1)/(d-1)$*, the average*

$$\overline{\operatorname{Avg}}(\phi,\beta,S):=\limsup_{B\to\infty}\frac{\sum_{h_K(f)\leqslant B}\#\big(\mathcal{O}_{\phi(f)}(\beta_f)\cap\mathcal{O}_{K,S}\big)}{\#\{f\in K\setminus k\mid h(f)\leqslant B\}}=0. \tag{55}$$

(Proof of Corollary 4.1). Let $k/\mathbb{F}_q$ be the constant field of K. We show that $I_{X,\phi,1}(K)=X(K)\setminus\mathbb{P}^1(k)$ and apply Theorem 4.1. Note first that $\phi(f)$ fixes $0,1,\infty$ for all $f\in K\setminus\{-1\}$. In particular, we deduce from [9, Proposition 12ii] that ϕ_f is isotrivial if and only if $f\in k$. Moreover, we compute the formal derivative

$$\phi(f)'(x)=\frac{(f+1)\cdot x^{d-2}\cdot(x^d+df\cdot x)}{(x^{d-1}+f)^2}, \tag{56}$$

and deduce that $\phi(f)$ is separable for all $f\neq -1$. Finally, we compute the second iterate

$$\phi^2(f)(x)=\frac{g_f(x)}{j_f(x)}=\frac{(f+1)\cdot x^{d^2}}{(f+1)^{d-1}\cdot x^{d(d-1)}\cdot(x^{d-1}+f)+f\cdot(x^{d-1}+f)^d} \tag{57}$$

and see that $\deg(g_f)=d^2$ and $\deg(j_f)=d^2-1$, whenever $f\neq -1$. In particular, $\phi^2(f)\in\bar{K}[x]$ implies that $g_f=j_f\cdot(ax+b)$ for some $a,b\in\bar{K}$. However, $g_f(0)=0$ implies that $f\cdot b=0$. Suppose first that $f\neq 0$. Then $b=0$ implies that the linear term $a\cdot f^{d+1}$ of $j_f(x)\cdot ax$ must be zero. This forces $a=0$, a contradiction. We conclude from (56) and (57) that $\phi(f)$ satisfies the hypothesis of [9, Theorem 2(ii)] for all $f\in\mathbb{P}^1(K)\setminus\mathbb{P}^1(k)$. In particular, $X(K)\setminus I_{X,\phi,m}(K)$ is finite and $\overline{\operatorname{Avg}}(\phi,\beta,S)=0$ by Theorem 4.1. ☐

Remark 4.4. In fact, one can show that $\big(\mathcal{O}_{\phi(f)}(b)\cap\mathcal{O}_{K,S}\big)$ is finite for all $f\neq 0,-1$ and $b\in K$. Hence, we could include $\phi(f)$ for $f\in\mathbb{P}^1(k)\setminus\{0,-1,\infty\}$ in our average; see the full statement of [9, Theorem 12], in particular, the case when $f\in k$. However, this would add only finitely many terms to the sum on (53) and would not affect the overall average.

(Proof of Theorem 4.1). We prove this result by applying the same techniques used in earlier theorems. Firstly, the height bounds in Proposition 2.1 translate verbatim to the global function field case, as do the bounds in the proof of Theorem 1.1 when $\phi:X\to\operatorname{Rat}_d$ and $\beta:X\to\mathbb{P}^1$ are separable (after we extend ϕ and β to morphisms from X to projective space). In particular, the degree condition $\deg(\beta)>(2d-1)/(d-1)\cdot\deg(\phi^*H_2)$ implies the crucial lower bound

$$\hat{h}_{\phi_P}(\beta_P)\geqslant\kappa_1\cdot h_{X,\beta^*H_1}(P)-\kappa_2, \tag{58}$$

for all $P \in X(K)$ such that: ϕ_P is defined, $d_s(\phi_P) \geqslant 2$, and $h_{X,\beta^*H_1}(P) > \delta_1$; here κ_1, κ_2 , and δ_1 are positive constants and H_1 and H_2 generate the hyperplane class in $\mathrm{Pic}(\mathbb{P}^1)$ and $\mathrm{Pic}(\mathbb{P}^{2d+1})$ respectively. In particular, the bound on (58) holds (and is positive) for all but finitely many $P \in X(K)$. Similarly, (8) implies that

$$h_{\mathbb{P}^{2d+1}}(\phi_P) \leqslant h_{X,\phi^*H_2}(P) + \kappa_3 \leqslant \kappa_4 \cdot h_{X,\beta^*H_1}(P) + \kappa_3 \tag{59}$$

for all $P \in X(K)$ such that $h_{X,\beta^*H_1}(P) \geqslant \delta_2$, where κ_3, κ_4, and δ_2 are positive constant.

Now to bound integral points. Suppose that $\phi_P^n(\beta_P) \in \mathcal{O}_{K,S}$ for some n. Then for all points $P \in I_{X,\phi,m}(K)$, the linear (and effective) height bound in [9, Theorem 2ii] implies that either $n \leqslant r_\phi := \min\{3m, m+4\}$ or

$$h_K\big(\phi_P^n(\beta_P)\big) \leqslant d^{r_\phi} \cdot q^{c_1} \cdot \big(2\mathfrak{g}_C - 2 + |S|\big) + c_2 \cdot h_{\mathbb{P}^{2d+1}}(\phi_P); \tag{60}$$

here $K = \mathbb{F}_q(C)$ for some curve $C_{/\mathbb{F}_q}$ of genus $\mathfrak{g}_C$, and c_1 and c_2 are absolute constants that depend on the inseparable degree of the extension $K(\phi^{-r}(\infty))/K$. Most importantly, all of the constants on (60) are independent of the point $P \in I_{X,\phi,m}(K)$; compare to Conjecture 3.1.

Suppose that $n > r$. Then parts (2) and (3) of Proposition 2.1 and the bounds on (58), (59), and (60) imply that

$$d^n \leqslant \frac{(\kappa_4 \cdot c_2) \cdot h_{X,\beta^*H_1}(P) + \big(d^{r_\phi} \cdot q^{c_1} \cdot \big(2\mathfrak{g}_C - 2 + |S|\big) + c_2 \cdot \kappa_3\big)}{\kappa_1 \cdot h_{X,\beta^*H_1}(P) - \kappa_2} \tag{61}$$

for all $P \in X(K)$ such that $h_{X,\beta^*H_1}(P) \geqslant \max\{\delta_1, \delta_2\}$. We obtain the familiar estimate that d^n is bounded by a linear fractional transformation in the height $h_{X,\beta^*H_1}(P)$. In particular, there exists an $N(\phi,\beta,S)$ such that $\phi_P^n(\beta_P) \in \mathcal{O}_{K,S}$ implies that $n \leqslant N(\phi,\beta,S)$ for all but finitely many $P \in I_{X,\phi,m}(K)$. On the other hand, since $(\mathcal{O}_{\phi_P}(\beta_P) \cap \mathcal{O}_{K,S})$ is finite for all points $P \in I_{X,\phi,m}(K)$ by [9, Theorem 2], we see that there is a maximum iterate that can produce an S-integral point for the finitely many exceptional points: $h_{X,\beta^*H_1}(P) \leqslant \max\{\delta_1, \delta_2\}$ and $\hat{h}_{\phi_P}(\beta_P) \neq 0$. We deduce that

$$\big\{n \mid P \in I_{X,\phi,m}(K),\ \phi_P^n(\beta_P) \in \mathcal{O}_{K,S},\ \text{and}\ \hat{h}_{\phi_P}(\beta_P) \neq 0\big\} \tag{62}$$

is bounded as claimed. On the other hand, (58) implies that

$$X_{\phi,\beta}^{\mathrm{PrePer}}(K) := \{P \in I_{X,\phi,m}(K) \mid \hat{h}_{\phi_P}(\beta_P) = 0\} \tag{63}$$

is finite. Hence, it follows from (62) and (63) that $\#\big(\mathcal{O}_{\phi_P}(\beta_P) \cap \mathcal{O}_{K,S}\big)$ is uniformly bounded over all $P \in I_{X,\phi,m}(K)$, since we may bound separately the number of S-integral points in the orbits of $P \in X_{\phi,\beta}^{\mathrm{PrePer}}(K)$ and use (62) for the wandering points.

Finally, suppose that $X = \mathbb{P}^1$. Then the density claim in Lemma 2.2, which translates to global function fields, implies that $\overline{\mathrm{Avg}}(\phi,\beta,S) = 0$; here we use that $I_{X,\phi.m}(K)$ and $X(K)$ have the same number of points asymptotically. □

Remark 4.5. Because of its independent interest, we note that over globals fields K and families (X,ϕ,S) as in Theorems 1.1 and 4.1, the quantity

$$\hat{h}_{X,\phi,\beta}^{\min} := \inf\big\{\hat{h}_{\phi_P}(\beta_P) \,\big|\, P \in X(K),\ \phi_P \text{ is defined}\big\} \tag{64}$$

is strictly positive whenever $X(K)$ is infinite; compare to (16) in the proof of Theorem 1.2. This follows from the height lower bounds on (10) and (58) respectively. In particular, these families share many of the same dynamical properties as constant families: $\phi : X \to \mathrm{Rat}_d$ such that $\phi_P = \varphi$ for all $P \in I_{X,\phi}(K)$.

As for future work in the number field case, there are several questions left to answer. First and foremost, one would like to prove Theorem 3.1 unconditionally. Similarly, since we know that $\overline{\mathrm{Avg}}_{\mathbb{Z}}(\phi, \beta, \mathbb{Z})$ is bounded in the 3-dimensional example in Theorem 3.2, it would be interesting to know if the average is zero in this particular family, as was the case in Theorems 1.2, 2.1, and 3.1. To do this however, one would need a maximum iterate $N(\phi, \beta)$ producing an integral point (see Corollary 3.1), as well as a suitable generalization of the density estimate in Lemma 2.2.

Furthermore, since the average in Theorem 3.2 makes sense for all rational specializations by Lemma 3.1, we would like to remove the integrality assumption. As a first step, it is likely that the canonical height lower bound on (41) holds for all $r, s, t \in \mathbb{Q}$ which are coprime in a suitable sense, implying that we have a uniform bound on the number of integral points for a positive proportion of terms in the average, similar to the $U(\mathbb{Z})$ case above.

Likewise, it would be interesting to know if the techniques in this paper generalize to families of rational maps of higher dimensional projective space. Although work on integral points in orbits of rational maps $\phi : \mathbb{P}^n \to \mathbb{P}^n$ is still in its infancy, there are scattered results, especially if one assumes the Vojta conjecture [34].

On the other hand, in a certain sense, studying families of rational maps of higher dimensional projective space is a more natural generalization of our work than that discussed in Section 3. For one, when X is a variety of dimension at least 2, the existence of a morphism $\beta : X \to \mathbb{P}^1$ implies that X has Picard rank at least 2. In particular, X cannot be projective space. On the other hand, suppose that X has large dimension and let

$$\mathrm{Rat}_d^n := \{\varphi : \mathbb{P}^n \to \mathbb{P}^n \mid \deg(\varphi) = d,\ \varphi \text{ is morphism}\}. \tag{65}$$

Then we have a much better chance of constructing morphisms $\phi : X \to \mathbb{P}^n$ and rational maps $\phi : X \to \mathrm{Rat}_d^n$ if we allow n to also be large. In particular, a first step would be to find conditions on $\deg(\beta^* H_1)$ and $\deg(\phi^* H_2)$ that imply a linear lower bound on $\hat{h}_{\phi_P}(\beta_P)$ as on (10).

Finally, a natural place to try and generalize our results is to study families of morphisms of $\mathbb{P}^1 \times \mathbb{P}^1$ of the form (φ, φ), where the finiteness of integral points in orbits is known [3].

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